New Conformal Field Theories with Anomalous Dimensions

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ABSTRACT

We find a class of fixed point theory for 2- and 3-dimensional non-linear sigma models using Wilsonian renormalization group (WRG) approach. In 2-dimensional case, the fixed point theory is equivalent to the Witten's semi-infinite cigar model. In 3-dimensional case, the theory has one parameter which describes a marginal deformation from the infrared to ultraviolet fixed points of the CP^N model in the theory spaces.

1. Introduction

The Wilsonian renormalization group equation describes the variation of effective action $S[\Omega;t]$ when the cutoff energy scale Λ is changed to $\Lambda(\delta t) = \Lambda e^{-\delta t}$ in D dimensional field theory [1][2][3]:

$$\begin{aligned}\frac{d}{dt}S[\Omega;t] &= \frac{1}{2\delta t}\int_{p'} tr \ln\left(\frac{\delta^2 S}{\delta\Omega^i\delta\Omega^j}\right) - \frac{1}{2\delta t}\int_{p'}\int_{q'}\frac{\delta S}{\delta\Omega^i(p')}\left(\frac{\delta^2 S}{\delta\Omega^i(p')\delta\Omega^j(q')}\right)^{-1}\frac{\delta S}{\delta\Omega^j(q')} \\ &\quad + \left[D - \sum_{\Omega_i}\int_p \hat{\Omega}_i(p)\left(d_{\Omega_i} + \gamma_{\Omega_i} + \hat{p}^\mu\frac{\partial}{\partial\hat{p}^\mu}\right)\frac{\delta}{\delta\hat{\Omega}_i(p)}\right]\hat{S}, \qquad (1)\end{aligned}$$

where d_Ω and γ_Ω denote the canonical and anomalous dimensions of the field Ω. The caret indicates dimensionless quantities. The first and second terms in eq.(1) correspond to the one-loop and tree diagrams contributions, when internal of fields with high momentum $\Lambda(\delta t) < p < \Lambda$ lines are eliminated. The remaining terms come from the rescaling of fields to normalize the coefficient of the kinetic term to unity. We impose $\mathcal{N}=2$ supersymmetry on the action and consider only Kähler potential term to define the $\mathcal{N}=2$ supersymmetric nonlinear sigma model in two- and three- dimensions

$$\begin{aligned} S &= \int d^2\theta d^2\bar{\theta} d^D x K[\Phi,\Phi^\dagger] \\ &= \int d^D x\Big[g_{n\bar{m}}\left(\partial^\mu\varphi^n\partial_\mu\varphi^{*\bar{m}} + \frac{i}{2}\bar{\psi}^{\bar{m}}\sigma^\mu(D_\mu\psi)^n + \frac{i}{2}\psi^n\bar{\sigma}^\mu(D_\mu\bar{\psi})^{\bar{m}} + \bar{F}^{\bar{m}}F^n\right) \\ &\quad -\frac{1}{2}K_{,nm\bar{l}}\bar{F}^{\bar{l}}\psi^n\psi^m - \frac{1}{2}K_{,n\bar{m}\bar{l}}F^n\bar{\psi}^{\bar{m}}\bar{\psi}^{\bar{l}} + \frac{1}{4}K_{,nm\bar{k}\bar{l}}(\bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{l}})(\psi^n\psi^m)\Big], \qquad (2)\end{aligned}$$

where Φ^n denote chiral superfields, whose components fields are complex scalars $\varphi^n(x)$, Dirac fermions $\psi^n(x)$ and complex auxiliary fields $F^n(x)$. The Kähler metric of the target space $g_{i\bar{j}}$ is given by the Kähler potential $g_{i\bar{j}}(\varphi,\varphi^*) = \frac{\partial^2 K(\varphi,\varphi^*)}{\partial\varphi^i\partial\varphi^{*\bar{j}}}$.

From the WRG equation, the β function for the Kähler metric is given by [4]

$$\beta(g_{i\bar{j}}) \;=\; \frac{1}{2\pi}R_{i\bar{j}} + \gamma\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2g_{i\bar{j}}\Big] + d_\varphi\Big[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}}\Big]. \tag{3}$$

Note that our β function reduces to the Ricci tensor when the anomalous dimension of the fields vanishes. The second term, proportional to the anomalous dimension γ, which is not reparametrization invariant because of the renormalization condition of the fields breaks reparametrization invariance. Since the Kähler metric contain the infinite number of coupling constants, the above WRG equation is an infinite set of differential equations for these coupling constants.

2. Fixed points with $U(N)$ symmetry in two-dimensions

Let us derive the action of the conformal field theory corresponding to the fixed-point of the β function (3) for $d_\varphi = 0$. Since Ricci curvature $R_{i\bar{j}}$ is a second derivative of the metric $g_{i\bar{j}}$, the equation is a set of coupled partial differential equations, and is very difficult to solve in general. So we simplify the problem by assuming symmetry $\mathbf{U}(N)$ for the Kähler potential.

$$K[\varphi, \varphi^\dagger] \;=\; \sum_{n=1}^{\infty} g_n x^n \equiv f(x) \tag{4}$$

where x is the $\mathbf{U}(N)$ invariant combination $x \equiv \vec{\varphi}\cdot\vec{\varphi}^\dagger$ of the N components scalar fields $\vec{\varphi} = (\varphi^1, \varphi^2, \cdots, \varphi^N)$. The coefficients g_n play the role of an infinite number of coupling constants which depend on the cutoff scale t.

We substitute the metric and Ricci tensor, obtained by the above Kähler potential, for the β function (**??**), and find that the fixed point theory satisfies following differential equation

$$\frac{\partial}{\partial t}f' \;=\; \frac{1}{2\pi}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x) = 0, \tag{5}$$

where $f' = \frac{df}{dx}$.

We found that the solution of this equation has a free parameter corresponding to the anomalous dimension of the field. In fact, the solution is very simple when the target manifold is of complex one-dimension $N = 1$:

$$f' = \frac{1}{ax}\ln(1 + ax). \tag{6}$$

Here, $a \equiv -4\pi\gamma$. This f' gives the metric of the target space

$$g_{i\bar{j}} = f' + f''x = \frac{1}{1 + ax}. \tag{7}$$

Note that the metric has only one component, and the indices i and $\bar{j}$ is 1. The volume and the distance from the origin ($|x| = 0$) to infinity ($|x| = \infty$) of target spaces are

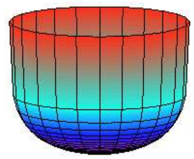

Figure 1: The target manifold embedded in 3-dimensional flat Euclidean spaces.

divergent, while the length of the circumference at the infinity is finite. Therefore, the shape of the target manifold is a semi-infinite cigar with radius $\sqrt{\frac{1}{a}}$.

This solution has been discussed in other context [5]. They consider the non-linear sigma model coupled with dilaton on curved space-time. The perturbative 1-loop β function for Kähler metric has the contribution of the dilaton field,

$$\beta(g_{i\bar{j}}) = R_{i\bar{j}} + 2\nabla_i \nabla_{\bar{j}} \Phi,$$

where Φ denotes the dilaton field. That last term plays the same role as the anomalous dimension term in (3). In fact, the above non-trivial dilaton gradient in space-time is equivalent to assigning a non-trivial Weyl transformation law to target space coordinates. Then, we can obtain the same fixed point theory using WRG equation.

3. Three-dimensional case

Next, we consider 3-dimensional non-linear sigma model. It is nonrenormalizable within the perturbative method, so we need non-perturbative analysis [6].

For example, we consider CP^N model which is defined by following Kähler potential:

$$K[\Phi, \Phi^\dagger] = \frac{1}{\lambda^2} \ln(1 + \Phi^i \Phi^{\dagger \bar{j}}). \tag{8}$$

where $i = 1, \cdots, N$. Using the β function (3) for $d_\varphi = \frac{1}{2}$, We obtain the anomalous dimension of scalar fields (or chiral superfields) and β function of the coupling constant λ as

$$\gamma \;=\; -\frac{(N+1)\lambda^2}{4\pi^2}, \qquad \beta(\lambda) = -\frac{N+1}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda. \tag{9}$$

We find there are an IR fixed point at $\lambda = 0$ and an UV fixed point at $\lambda^2 = \frac{2\pi^2}{N+1} \equiv \lambda_c^2$.

Now, we investigate the conformal field theories defined as the fixed point of the β function (3) for $d_\varphi = \frac{1}{2}$. To simplify, we assume $\mathbf{SU}(N)$ symmetric Kähler potential (4) as before. We substitute the metric and Ricci tensor for the β function (3) and obtain the following differential equation.

$$\frac{\partial}{\partial t} f' = \frac{1}{2\pi^2}\Big[(N-1)\frac{f''}{f'} + \frac{2f'' + f'''x}{f' + f''x}\Big] - 2\gamma(f' + f''x) - f''x = 0. \tag{10}$$

To obtain a conformal field theory, we must solve the differential equation The function $f(x)$ is a polynomial of infinite degree, and it is hard to solve it analytically. So we expand the function $f(x)$ and the equation (10) around $x \approx 0$. Then the following function satisfies $\beta = 0$ for any values of parameter g_2:

$$\begin{aligned} f' &= 1 + 2g_2 x + [\frac{2(3N+5)}{N+2} g_2^2 + \frac{2\pi^2}{N+2} g_2] x^2 \\ &+ \frac{4}{3(N+2)(N+3)} [(16N^2 + 66N + 62) g_2^3 + 2\pi^2 (6N+14) g_2^2 + 2\pi^4 g_2] x^3 + \cdots . \end{aligned} \quad (11)$$

The parammeter g_2 corresponds to the anomalous dimension of scalar field as in the 2-dimensional case. The function $f(x)$ describe the conformal field theory and has one free parameter g_2. Thus, if we fix the value of g_2, we obtain a conformal field theory.

For example, we take $g_2 = -\frac{1}{2} \cdot \frac{2\pi^2}{N+1} \equiv -\frac{1}{2} a$. Then, the function $f(x)$ becomes

$$f(x) = \frac{1}{a} \ln(1 + ax), \quad (12)$$

and this is the Kähler potential of CP^N model. Thus one of the novel $\mathbf{SU}(N)$ symmetric conformal field theory is equal to the UV fixed point theory of CP^N model for the specific value of g_2. In this case, the symmetry of this theory enhances to $\mathbf{SU}(N+1)$ because the CP^N model has the isometry $\mathbf{SU}(N+1)$.

4. Summary

We constructed a class of the $\mathbf{SU}(N)$ symmetric conformal field theory by using the WRG equation. This has one free parameter corresponding to the anomalous dimension of the scalar fields. If we choose a specific value of the parameter, we recover the conformal field theory defined at the UV fixed point of CP^N model and the symmetry is enhanced to $\mathbf{SU}(N+1)$.

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6. References

[1] K.G. Wilson and I.G. Kogut, Phys.Rep.**12** (1974) 75.
[2] F. Wegner and A. Houghton, Phys.Rev.**A8** (1973) 401.
[3] T.R. Morris, Int. J. Mod. Phys. A **9** (1994) 2411, `hep-ph/9308265`.
[4] K. Higashijima and E. Itou, Prog. Theor. Phys. **109** 751, `hep-th/0302090`.
[5] E. Witten, Phys. Rev. **D44** (1991) 314; E. Kiritsis, C. Kounnas and D. Lust, Int. J. Mod. Phys. **A9** (1994) 1361-1394; K. Hori and A. Kapustin, JHEP **0108** (2001) 045
[6] K. Higashijima and E. Itou, Prog. Theor. Phys. **110** 563, `hep-th/0304194`.